Topological Dynamics of Cellular Automata: Dimension Matters

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Abstract. Topological dynamics of cellular automata (CA), inherited from classical dynamical systems theory, has been essentially studied in dimension 1. This paper focuses on higher dimensional CA and aims at showing that the situation is different and more complex starting from dimension 2. The main results are the existence of non sensitive CA without equicontinuous points, the non-recursivity of sensitivity constants, the existence of CA having only non-recursive equicontinuous points and the existence of CA having only countably many equicontinuous points. They all show a difference between dimension 1 and higher dimensions. Thanks to these new constructions, we also extend undecidability results concerning topological classification previously obtained in the 1D case. Finally, we show that the set of sensitive CA is only Π_2^0 in dimension 1, but becomes Σ_3^0-hard for dimension 3.

1 Introduction

Cellular automata were introduced by J. von Neumann as a simple formal model of cellular growth and replication. They consist in a discrete lattice of finite-state machines, called *cells*, which evolve uniformly and synchronously according to a local rule depending only on a finite number of neighbouring cells. A snapshot of the states of the cells at some time of the evolution is called a *configuration*, and a cellular automaton can be view as a global action on the set of configurations.

Despite the apparent simplicity of their definition, cellular automata can have very complex behaviours. One way to try to understand this complexity is to endow the space of configurations with a topology and consider cellular automata as classical dynamical systems. With such a point of view, one can use well-tried tools from dynamical system theory like the notion of sensitivity to initial condition or the notion of equicontinuous point.

This approach has been followed essentially in the case of one-dimensional cellular automata. P. Kůrka has shown in [2] that 1D cellular automata are partitioned into two classes:

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– $\mathcal{E}_{qu}$, the set of cellular automata with equicontinuous points,
– $\mathcal{S}_{ens}$, the set of sensitive cellular automata.

We stress that this partition result is false in general for classical (continuous) dynamical systems. Thus, it is natural to ask whether this result holds for the model of CA in any dimension, or if it is a "miracle" or an "anomaly" of the one-dimensional case due to the strong constraints on information propagation in this particular setting. One of the main contributions of this paper is to show that this is an anomaly of the 1D case (Section 3): there exist a class $\mathcal{N}$ of 2D CA which are neither in $\mathcal{E}_{qu}$ nor in $\mathcal{S}_{ens}$.

Each of the sets $\mathcal{E}_{qu}$ and $\mathcal{S}_{ens}$ has an extremal sub-class: equicontinous and expansive cellular automata (respectively). This allows to classify cellular automata in four classes according to the degree of sensitivity to initial conditions. The dynamical properties involved in this classification have been intensively studied in the literature for 1D cellular automata (see for instance [2,3,4,5]). Moreover, in [6], the undecidability of this classification is proved, except for the expansivity class whose decidability remains an open problem.

In this paper, we focus on 2D CA and we are particularly interested in differences from the 1D case. As said above, we will prove in Section 3 that there is a fundamental difference with respect to the topological dynamics classification, but we will also adopt a computational complexity point of view and show that some properties or parameters which are computable in 1D are non recursive in 2D (Proposition 8 and 9 of Section 5). To our knowledge, only few dimension-sensitive undecidability results are known for CA ([7,8]). However, we believe that such subtle differences are of great importance in a field where the common belief is that everything interesting is undecidable.

Moreover, we establish in Section 5 several complexity lower bounds on the classes defined above and extend the undecidability result of [6] to dimension 2. Notably, we show that each of the class $\mathcal{E}_{qu}$, $\mathcal{S}_{ens}$ and $\mathcal{N}$ is neither recursively enumerable, nor co-recursively enumerable. This gives new examples of "natural" properties of CA that are harder than the classical problems like reversibility, surjectivity or nilpotency (which are all r.e. or co-r.e.).

Finally, we show two additional results advocating the importance of dimension in topological dynamics: first, there are 2D CA having only a countable set of equicontinuous points and, second, the set of sensitive CA raises from Π_2^0 in dimension 1 to Σ_3^0-complete in dimension 3.

2 Definitions

Let $\mathcal{A}$ be a finite set and $\mathbb{M} = \mathbb{Z}^d$ (for the d-dimensional case). We consider $\mathcal{A}^{\mathbb{M}}$, the *configuration space* of $\mathbb{M}$-indexed sequences in $\mathcal{A}$.

If $\mathcal{A}$ is endowed with the discrete topology, $\mathcal{A}^{\mathbb{M}}$ is compact, perfect and totally disconnected in the product topology. Moreover one can define a metric on $\mathcal{A}^{\mathbb{M}}$ compatible with this topology:

$$\forall x, y \in \mathcal{A}^{\mathbb{M}}, \quad d_C(x, y) = 2^{-\min\{\|i\|_\infty : x_i \neq y_i \ i \in \mathbb{M}\}}.$$

Let $\mathbb{U} \subset \mathbb{M}$. For $x \in \mathcal{A}^{\mathbb{M}}$, denote $x_{\mathbb{U}} \in \mathcal{A}^{\mathbb{U}}$ the restriction of x to $\mathbb{U}$. Let $\mathbb{U} \subset \mathbb{M}$ be a finite subset, Σ is a *subshift of finite type of order* $\mathbb{U}$ if there exists $\mathcal{F} \subset \mathcal{A}^{\mathbb{U}}$ such that $x \in \Sigma \iff x_{m+\mathbb{U}} \in \mathcal{F} \quad \forall m \in \mathbb{M}$. In other word, Σ can be viewed as a tiling where the allowed patterns are in $\mathcal{F}$.

In this paper, we will consider *tile sets* and ask whether they can tile the plane or not. In our formalism, a tile set is a subshift of finite type: a set of states (the tiles) given together with a set of allowed patterns (the tiling constraints).

A *cellular automaton* (CA) is a pair $(\mathcal{A}^{\mathbb{M}}, F)$ where $F : \mathcal{A}^{\mathbb{M}} \to \mathcal{A}^{\mathbb{M}}$ is defined by $F(x)(m) = f((x(m+u))_{u\in\mathbb{U}})$ for all $x \in \mathcal{A}^{\mathbb{M}}$ and $m \in \mathbb{M}$ where $\mathbb{U} \subset \mathbb{Z}$ is a finite set named *neighbourhood* and $f : \mathcal{A}^{\mathbb{U}} \to \mathcal{A}$ is a *local rule*. The radius of F is $r(F) = \max\{\|u\|_\infty : u \in \mathbb{U}\}$. By Hedlund's theorem [9], it is equivalent to say that F is a continuous function which commutes with the shift (i.e. $\sigma^m \circ F = F \circ \sigma^m$ for all $m \in \mathbb{M}$).

We recall here general definitions of topological dynamics used all along the article. Let (X, d) be a metric space and $F : X \to X$ be a continuous function.

• $x \in X$ is an *equicontinuous point* if for all $\varepsilon > 0$, there exists $\delta > 0$, such that for all $y \in X$, if $d(x, y) < \delta$ then $d(F^n(x), F^n(y)) < \varepsilon$ for all $n \in \mathbb{N}$.

• (X, F) is *sensitive* if there exists $\varepsilon > 0$ such that for all $\delta > 0$ and $x \in X$, there exists $y \in X$ and $n \in \mathbb{N}$ such that $d(x, y) < \delta$ and $d(F^n(x), F^n(y)) > \varepsilon$.

In the definition above about properties of topological dynamics, the dimension of the cellular automaton considered do not appear explicitly. Whereas essentially studied in dimension 1 in the literature, the present paper consider those properties in any dimension. A first (trivial) approach to study topological dynamics properties according to dimension is given by the following proposition through the notion of canonical lift from dimension d to dimension $d + 1$. The canonical lift of a CA of dimension d with neighbourhood $\mathbb{U}$ and local rule f is the CA of dimension $d+1$, of local rule f and of neighbourhood $\mathbb{U}'$ obtained by adding a coordinate equal to 0 to each vector of $\mathbb{U}$.

Proposition 1. *Let F be a CA of dimension d and let $F^\uparrow$ be its canonical lift to dimension $d + 1$. Then we have the following:*

- *F has equicontinuous points if and only if $F^\uparrow$ has equicontinuous points;*
- *F is sensitive to initial conditions if and only if $F^\uparrow$ is sensitive to initial conditions.*

Proof. Straightforward. □

This proposition essentially says that what can be "seen" in dimension d (concerning some topological dynamics properties) can also be "seen" in dimension $d+1$. One of the main point of the present paper is to show that the converse is false: some behaviours cannot be "seen" in low-dimensional cellular automata.

3 The Core Construction

In this section, we will construct a 2D CA which has no equicontinuous point and is not sensitive to initial conditions. This is in contrast with dimension 1 where any non-sensitive CA must have equicontinuous points as shown in [2] (such differences according to dimension will be further discussed in Section 5).

The CA (denoted by F in the following) is made of two components:

- a *solid component* (almost static) for which only finite type conditions are checked and corrections are made locally ;
- a *liquid component* whose overall behaviour is to infiltrate the solid component and allow some particles to move left and to bypass solid obstacles.

The general behaviour of this cellular automaton can be seen as an erosion/infiltration process. States from the solid component can be turned into liquid state according to certain local conditions but the converse is impossible. Therefore the set of solid states is decreasing (erosion process) until some particular kind of configuration is reached (erosion result). Then, in such configurations, the particles can bypass any sequence of obstacles and reach any liquid position (infiltration).

3.1 Definition

Formally, F has a Moore's neighbourhood of radius 2 (25 neighbours) and a state set $\mathcal{A}$ with 12 elements : $\mathcal{A} = \{U, D, 0, 1, \downarrow, \uparrow, \leftarrow, \rightarrow, \swarrow, \searrow, \nwarrow, \nearrow\}$ where the subset $\mathcal{S} = \{1, \downarrow, \uparrow, \leftarrow, \rightarrow, \swarrow, \searrow, \nwarrow, \nearrow\}$ corresponds to the solid component and $\mathcal{L} = \{U, D, 0\}$ to the liquid component where 0 should be thought as the substratum where particles made of elementary constituents U and D can move.

Let $\Sigma_{\mathcal{S}}$ be the subshift of finite type of $\mathcal{A}^{\mathbb{Z}^2}$ defined by the set of allowed patterns constituted by all the 3×3 patterns appearing in the following set of finite configurations:

$$\begin{array}{cccccccccc}
\mathcal{L} & \mathcal{L} & \mathcal{L} & \mathcal{L} & \mathcal{L} & \mathcal{L} & \mathcal{L} & \mathcal{L} & \mathcal{L} & \mathcal{L} \\
\mathcal{L} & \mathcal{L} & \mathcal{L} & \mathcal{L} & \mathcal{L} & \mathcal{L} & \mathcal{L} & \mathcal{L} & \mathcal{L} & \mathcal{L} \\
\mathcal{L} & \mathcal{L} & \mathcal{L} & \searrow & \downarrow & \downarrow & \downarrow & \swarrow & \mathcal{L} & \mathcal{L} \\
\mathcal{L} & \mathcal{L} & \mathcal{L} & \rightarrow & 1 & 1 & 1 & \leftarrow & \mathcal{L} & \mathcal{L} \\
\mathcal{L} & \mathcal{L} & \mathcal{L} & \rightarrow & 1 & 1 & 1 & \leftarrow & \mathcal{L} & \mathcal{L} \\
\mathcal{L} & \mathcal{L} & \mathcal{L} & \rightarrow & 1 & 1 & 1 & \leftarrow & \mathcal{L} & \mathcal{L} \\
\mathcal{L} & \mathcal{L} & \mathcal{L} & \nearrow & \uparrow & \uparrow & \uparrow & \nwarrow & \mathcal{L} & \mathcal{L} \\
\mathcal{L} & \mathcal{L} & \mathcal{L} & \mathcal{L} & \mathcal{L} & \mathcal{L} & \mathcal{L} & \mathcal{L} & \mathcal{L} & \mathcal{L} \\
\mathcal{L} & \mathcal{L} & \mathcal{L} & \mathcal{L} & \mathcal{L} & \mathcal{L} & \mathcal{L} & \mathcal{L} & \mathcal{L} & \mathcal{L}
\end{array}$$

Intuitively, $\Sigma_{\mathcal{S}}$ defines the 'admissible' solid obstacles, *i.e.* solid shapes that are stable and no longer eroded in a liquid environment.

The local transition function of F can be sketched as follows:

- states from $\mathcal{S}$ are turned into 0's if finite type conditions defining $\Sigma_{\mathcal{S}}$ are violated locally and left unchanged in any other case ;
- states U and D behave like a left-moving particle when U is just above D in a background of 0's, and they separate to bypass solid obstacles, U going over and D going under, until they meet at the opposite position and recompose a left-moving particle (see Figure 1).

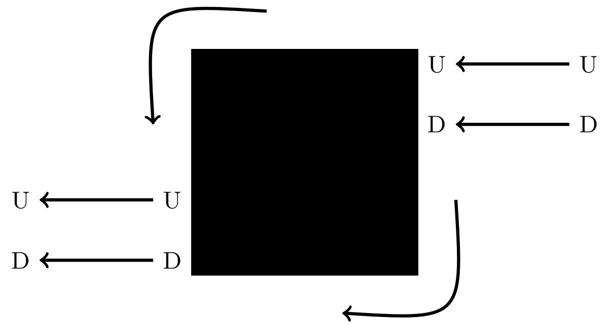

Fig. 1. A particle separating into two parts (U and D) to bypass a solid obstacle (the black region).

A precise definition of the local transition function of F is the following:

1. if the neighbourhood (5×5 cells) forms a pattern forbidden in $\Sigma_{\mathcal{S}}$, then turn into state 0 ;
2. else, apply (if possible) one of the transition rules depending only on the 3×3 neighbourhood detailed in Figure 2;
3. in any other case, turn into state 0.

Note for instance, that any solid state surrounded by a valid neighbourhood is left unchanged by F (second case of the definition above apply since the 3 first transitions of Figure 2 include all possible valid 3×3 neighbourhoods seen by a solid state).

3.2 Erosion and Infiltration

A configuration x is said to be *finite* if the set $\{z : x(z) \neq 0\}$ is finite. The next lemma shows that $\Sigma_{\mathcal{S}}$ attracts any finite configuration under the action of F. Moreover, after some time, all particles are on the left of the finite solid part.

Lemma 1 (erosion process). *For any finite configuration x, there exists t_0 such that $\forall t \geq t_0 : F^t(x) \in \Sigma_{\mathcal{S}}$ and, in $F^t(x)$, any occurrence of U or D is on the left of any occurrence of any state from $\mathcal{S}$.*

Proof. First, the set $\{z : x(z) \in \mathcal{S}\}$ is finite and decreasing under the action of F. Moreover, U and D states can only move left, or move vertically or disappear. Since the total amount of vertical moves for U and D states is bounded by the cardinal of $\{z : x(z) \in \mathcal{S}\}$, there is a time t after which all U or D state are on the left of all occurrences of states from $\mathcal{S}$, and each U is above a D in a 0 background (the UD particle is on the left of the finite non-0 region). From this time on, the evolution of cells in a state of $\mathcal{S}$ is governed only by the first case of the definition of F. Therefore, after a certain time, finite type conditions defining $\Sigma_{\mathcal{S}}$ are verified everywhere. To conclude, it is easy to check that $\Sigma_{\mathcal{S}}$ is stable under the action of F. $\square$

$$\begin{array}{c|c|c} \mathcal{L} & \mathcal{L} & \mathcal{L} \\ \hline \mathcal{S} & x \in \mathcal{S} & \mathcal{L} \\ \hline \mathcal{S} & \mathcal{S} & \mathcal{L} \end{array} \mapsto x, \qquad \begin{array}{c|c|c} \mathcal{S} & \mathcal{S} & \mathcal{L} \\ \hline \mathcal{S} & x \in \mathcal{S} & \mathcal{L} \\ \hline \mathcal{S} & \mathcal{S} & \mathcal{L} \end{array} \mapsto x, \qquad \begin{array}{c|c|c} \mathcal{S} & \mathcal{S} & \mathcal{S} \\ \hline \mathcal{S} & x \in \mathcal{S} & \mathcal{S} \\ \hline \mathcal{S} & \mathcal{S} & \mathcal{S} \end{array} \mapsto x,$$

$$\begin{array}{c|c|c} 0 \text{ or } \mathcal{S} & 0 & 0 \\ \hline \mathcal{S} & 0 & 0 \\ \hline \mathcal{S} & U & 0 \end{array} \mapsto U, \qquad \begin{array}{c|c|c} 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline \mathcal{S} & U & 0 \end{array} \mapsto U, \qquad \begin{array}{c|c|c} 0 & 0 & 0 \\ \hline 0 & 0 & U \\ \hline \mathcal{S} & \mathcal{S} & D \text{ or } \mathcal{S} \end{array} \mapsto U,$$

$$\begin{array}{c|c|c} 0 & 0 & 0 \\ \hline 0 & 0 & U \\ \hline 0 & 0 \text{ or } \mathcal{S} & \mathcal{S} \end{array} \mapsto U, \qquad \begin{array}{c|c|c} 0 & U & 0 \text{ or } \mathcal{S} \\ \hline 0 & 0 & \mathcal{S} \\ \hline 0 & 0 & \mathcal{S} \end{array} \mapsto U, \qquad \begin{array}{c|c|c} 0 & U & \mathcal{S} \\ \hline 0 & 0 & \mathcal{S} \\ \hline 0 & 0 & D \end{array} \mapsto U,$$

$$\begin{array}{c|c|c} \mathcal{S} & D & 0 \\ \hline \mathcal{S} & 0 & 0 \\ \hline 0 \text{ or } \mathcal{S} & 0 & 0 \end{array} \mapsto D, \qquad \begin{array}{c|c|c} \mathcal{S} & D & 0 \\ \hline 0 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array} \mapsto D, \qquad \begin{array}{c|c|c} \mathcal{S} & \mathcal{S} & U \text{ or } \mathcal{S} \\ \hline 0 & 0 & D \\ \hline 0 & 0 & 0 \end{array} \mapsto D,$$

$$\begin{array}{c|c|c} 0 & 0 \text{ or } \mathcal{S} & \mathcal{S} \\ \hline 0 & 0 & D \\ \hline 0 & 0 & 0 \end{array} \mapsto D, \qquad \begin{array}{c|c|c} 0 & 0 & \mathcal{S} \\ \hline 0 & 0 & \mathcal{S} \\ \hline 0 & D & 0 \text{ or } \mathcal{S} \end{array} \mapsto D, \qquad \begin{array}{c|c|c} 0 & 0 & U \\ \hline 0 & 0 & \mathcal{S} \\ \hline 0 & D & \mathcal{S} \end{array} \mapsto D$$

$$\begin{array}{c|c|c} 0 \text{ or } \mathcal{S} & 0 \text{ or } \mathcal{S} & U \\ \hline 0 \text{ or } \mathcal{S} & 0 & D \\ \hline 0 \text{ or } \mathcal{S} & 0 \text{ or } \mathcal{S} & 0 \text{ or } \mathcal{S} \end{array} \mapsto D, \qquad \begin{array}{c|c|c} 0 \text{ or } \mathcal{S} & 0 \text{ or } \mathcal{S} & 0 \text{ or } \mathcal{S} \\ \hline 0 \text{ or } \mathcal{S} & 0 & U \\ \hline 0 \text{ or } \mathcal{S} & 0 \text{ or } \mathcal{S} & D \end{array} \mapsto U$$

Fig. 2. Part of the transition rule of F (curved arrows mean that the transition is the same for any rotation of the neighbourhood pattern by an angle multiple of $\pi/2$).

The following lemma states that finite configurations from $\Sigma_{\mathcal{S}}$ consist of rectangle obstacles inside a liquid background. Moreover, obstacles are spaced enough to ensure that any position "sees" at most one obstacle in its 3×3 neighbourhood.

In the sequel we use notation $\texttt{South}(\cdot)$, $\texttt{East}(\cdot)$, $\texttt{West}(\cdot)$, $\texttt{North}(\cdot)$ for the elementary translations in $\mathbb{Z}^2$.

Lemma 2 (erosion result). *Let $x \in \Sigma_{\mathcal{S}}$ be a finite configuration. Then the set $X = \{z \in \mathbb{Z}^2 : x(z) \in \mathcal{S}\}$ is a union of disjoint rectangles which are pairwise spaced by at least 2 cells.*

Proof. Straightforward from definition of $\Sigma_{\mathcal{S}}$. □

An *obstacle* is a (finite) rectangular region of states from $\mathcal{S}$ surrounded by liquid states.

The following lemma establishes the key property of the dynamics of F: particles can reach any liquid position inside a finite field of obstacles from arbitrarily far away from the field.

Lemma 3 (infiltration). *Let $x \in \Sigma_{\mathcal{S}}$ be a finite configuration. For any $z_0 \in \mathbb{Z}^2$ such that $x(z_0) = 0$ there exists a path (z_n) such that:*

1. $\|z_n\|_\infty \to \infty$
2. $\exists n_0, \forall n \geq n_0$, *if x_n is the configuration obtained from x by adding a particle at position z_n (precisely, $x_n(z_n) = U$ and $x_n\big(\texttt{South}(z_n)\big) = D$) then $\big(F^n(x_n)\big)(z_0) \in \{U, D\}$.*

Proof. First, we suppose that $x \in \Sigma_{\mathcal{S}} \cap \big(\{0\} \cup \mathcal{S}\big)^{\mathbb{Z}^2}$. Since $x \in \Sigma_{\mathcal{S}}$ and $x(z_0) = 0$, then either $x\big(\texttt{South}(z_0)\big) = 0$ or $x\big(\texttt{North}(z_0)\big) = 0$ by Lemma 2. We will consider only the first case since the proof for the second one is similar. Let (z_n) be the path starting from z_0 defined as follows:

– If $x\big(\texttt{East}(z_n)\big) = 0$ and $x\big(\texttt{South}(\texttt{East}(z_n))\big) = 0$ then $z_{n+1} = \texttt{East}(z_n)$.
– Else, position $\texttt{East}(z_n)$ and/or position $\texttt{South}(\texttt{East}(z_n))$ belongs to an obstacle P. Let a, b and c be the positions of the upper-left, upper-right and lower-right outside corners of P and let p be its half perimeter. Then define $z_{n+1}, \ldots, z_{n+p+1}$ to be the sequence of positions made of (see Figure 3):
  - a (possibly empty) vertical segment from z_n to a,
  - the segment $[a; b]$,
  - a (possibly empty) vertical segment from b to z_{n+p+1} where z_{n+p+1} is the point on $[b; c]$ such that $z_n a + b z_{n+p+1} = bc$.

We claim that the path (z_n) constructed above has the properties of the lemma. Indeed, one can check that for each case of the inductive construction of a point z_m from a point z_n we have:

– $\|z_m\|_\infty > \|z_n\|_\infty$,
– $F^{m-n}(x_m)(z_n) = U$ and $F^{m-n}(x_m)(\texttt{South}(z_n)) = D$.

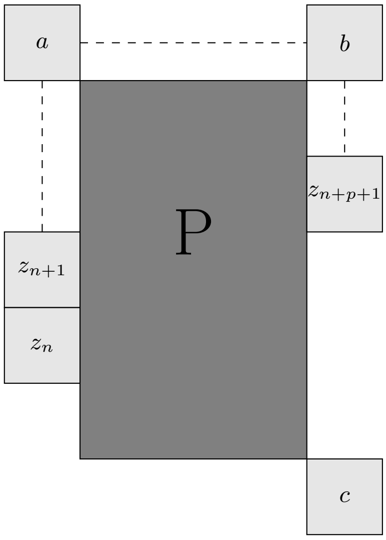

Fig. 3. Definition of the path $(z_n)_n$ in the presence of obstacles.

The lemma is thus proved for $x \in \Sigma_{\mathcal{S}} \cap \big(\{0\} \cup \mathcal{S}\big)^{\mathbb{Z}^2}$. It extends to any finite $x \in \Sigma_{\mathcal{S}}$ because in such a configuration Lemma 1 ensures that after some time t_0 all occurrences of U and D are on the left of z_0, whereas the path constructed above is on the right of z_0. More precisely, if x' is the configuration obtained from x by replacing any liquid state by 0, and if $(z_n)_n$ is the path constructed for x', then the path $(z_{t_0+n})_n$ fulfils the requirements of the lemma for x. □

3.3 Topological Dynamics Properties

The possibility to form arbitrarily large obstacles prevents F from being sensitive to initial conditions.

Proposition 2. *F is not sensitive to initial conditions.*

Proof. Let $\varepsilon > 0$. Let c_ε be the configuration everywhere equal to 0 except in the square region of side $2\lceil -\log \varepsilon \rceil$ around the center where there is a valid obstacle. $\forall y \in \mathcal{A}^{\mathbb{Z}^2}$, if $d(y, c_\varepsilon) \leq \varepsilon/4$ then $\forall t \geq 0$, $d\big(F^t(c_\varepsilon), F^t(y)\big) \leq \varepsilon$ since a well-formed obstacle (precisely, a partial configuration that would form a valid obstacle when completed by 0 everywhere) is unalterable for F provided it is surrounded by states in $\mathcal{L}$ (see the 3 first transition rules of case 2 in the definition of the local rule): this is guarantied for y by the condition $d(y, c_\varepsilon) \leq \varepsilon/4$. □

The erosion and infiltration process described above ensures that particles can circulate everywhere in the liquid part of finite configurations. This is the key ingredient of the following proposition.

Proposition 3. *F has no equicontinuous points.*

Proof. Assume F has an equicontinuous point, precisely a point x which verifies $\forall\varepsilon > 0, \exists\delta : \forall y, d(x, y) \le \delta \Rightarrow \forall t, d\big(F^t(x), F^t(y)\big) \le \varepsilon$.

Suppose that there is z_0 such that $x(z_0) = 0$ and let $\varepsilon = 2^{-\|z_0\|_\infty - 1}$. We will show that the hypothesis of x being an equicontinuous point is violated for this particular choice of ε. Consider any $\delta > 0$ and let y be the configuration everywhere equal to 0 except in the central region of radius $-\log\lceil\delta\rceil$ where it is identical to x. Since y is finite, there exists t_0 such that $y_+ = F^{t_0}(y) \in \Sigma_{\mathcal{S}}$ (by Lemma 1). Moreover, Lemma 1 guaranties that for any positive integer t, $F^t(y_+)(z_0) = x(z_0) = 0$. Applying Lemma 3 on y_+ and position z_0, we get the existence of a path (z_n) allowing particles placed arbitrarily far away from z_0 to reach the position z_0 after a certain time. For any sufficiently large n, we construct a configuration y' obtained from y by adding a particle at position z_n. By the property of (z_n), we have: $F^n(y)(z_0) \ne F^n(y')(z_0)$ and therefore $d\big(F^n(y), F^n(y')\big) > \varepsilon$. Since, if $n > -\log\lceil\delta\rceil$, both y and y' are in the ball of center x and radius δ, we have the desired contradiction.

Assume now that $\forall z, x(z) \in \mathcal{S}$. There must exist some position z_0 such that $x(z_0) \in \mathcal{S} \setminus \{1\}$ (it is straightforward to check that the uniform configuration everywhere equal to 1 is not an equicontinuous point). It follows from the definition of $\Sigma_{\mathcal{S}}$ that z_0 belongs to a forbidden pattern for $\Sigma_{\mathcal{S}}$ (any solid state different from 1 must have a liquid state in its neighbourhood). Therefore $F(x)(z_0) = 0$ and we can use the reasoning of the previous case of this proof on configuration $F(x)$.

Finally, if $\forall z, x(z) \ne 0$ and $\exists z_0, x(z_0) \in \{U, D\}$ then necessarily $F(x)(z_0) = 0$ and the first reasoning of the proof can be applied. $\square$

4 Variations

4.1 Adding Wang Tile Constraints

The first variation on F we consider is to add some tiling constraints to the solid component.

More precisely, for any tile set τ, we define a 2D CA F_τ which is identical to F except for the following modifications:

- the solid state 1 is replaced by the set τ so that the state set of F_τ is $\mathcal{A}_\tau = \big\{U, D, 0, \downarrow, \uparrow, \leftarrow, \rightarrow, \swarrow, \searrow, \nwarrow, \nearrow\big\} \cup \tau$ where the solid component is the subset $\mathcal{S}_\tau = \{\downarrow, \uparrow, \leftarrow, \rightarrow, \swarrow, \searrow, \nwarrow, \nearrow\} \cup \tau$ and the liquid component is also $\mathcal{L} = \{U, D, 0\}$;
- the sub-shift of 'admissible' obstacles now becomes $\Sigma_{F,\tau}$ defined by the set of allowed patterns constituted by all the 3×3 patterns appearing in the following set of finite configurations:

$$\begin{array}{cccccccccc}
\mathcal{L} & \mathcal{L} & \mathcal{L} & \mathcal{L} & \mathcal{L} & \mathcal{L} & \mathcal{L} & \mathcal{L} & \mathcal{L} & \mathcal{L} \\
\mathcal{L} & \mathcal{L} & \mathcal{L} & \mathcal{L} & \mathcal{L} & \mathcal{L} & \mathcal{L} & \mathcal{L} & \mathcal{L} & \mathcal{L} \\
\mathcal{L} & \mathcal{L} & \mathcal{L} & \searrow & \downarrow & \downarrow & \downarrow & \swarrow & \mathcal{L} & \mathcal{L} \\
\mathcal{L} & \mathcal{L} & \mathcal{L} & \rightarrow & \tau & \tau & \tau & \leftarrow & \mathcal{L} & \mathcal{L} \\
\mathcal{L} & \mathcal{L} & \mathcal{L} & \rightarrow & \tau & \tau & \tau & \leftarrow & \mathcal{L} & \mathcal{L} \\
\mathcal{L} & \mathcal{L} & \mathcal{L} & \rightarrow & \tau & \tau & \tau & \leftarrow & \mathcal{L} & \mathcal{L} \\
\mathcal{L} & \mathcal{L} & \mathcal{L} & \nearrow & \uparrow & \uparrow & \uparrow & \nwarrow & \mathcal{L} & \mathcal{L} \\
\mathcal{L} & \mathcal{L} & \mathcal{L} & \mathcal{L} & \mathcal{L} & \mathcal{L} & \mathcal{L} & \mathcal{L} & \mathcal{L} & \mathcal{L} \\
\mathcal{L} & \mathcal{L} & \mathcal{L} & \mathcal{L} & \mathcal{L} & \mathcal{L} & \mathcal{L} & \mathcal{L} & \mathcal{L} & \mathcal{L}
\end{array}$$

with the additional condition that two adjacent cells in a state from τ must fulfils the tiling constraints involved in the tile set τ.

The behaviour of F_τ is similar to that of F replacing $\Sigma_{\mathcal{S}}$ by $\Sigma_{F,\tau}$. More precisely:

1. if the neighbourhood (5×5 cells) forms a pattern forbidden in $\Sigma_{F,\tau}$, then turn into state 0;
2. else, apply (if possible) one of the transition rules depending only on the 3×3 neighbourhood detailed in Figure 2 (replacing $\mathcal{S}$ by $\mathcal{S}_\tau$);
3. in any other case, turn into state 0.

As for F, the erosion/infiltration mechanism prevents from any equicontinuous point. Moreover the sensitivity to initial conditions of F_τ is controlled by the tile set τ as shown by the following proposition.

Proposition 4. *Let τ be any tile set. Then we have the following:*

- *F_τ has no equicontinuous point;*
- *F_τ is sensitive to initial conditions if and only if τ does not tile the plane. Moreover, in this case, the maximal sensitivity constant is an exponential function of n, where $n \times n$ is the size of the largest admissible square tiling.*

Proof. Firstly, it follows from definition of F_τ that Lemmas 1, 2 and 3 as well as Proposition 3 remain true. Indeed, considering any configuration x of F_τ, and any $t \geq 0$, then we have

$$\{z : F_\tau^t(x)(z) \in \mathcal{S}_\tau\} \subseteq \{z : F^t(x')(z) \in \mathcal{S}\}$$

where x' is the configuration of F obtained from x be replacing any occurrence of states from τ by 1.

Moreover, if τ can tile the plane then it is possible to form arbitrarily large valid obstacles, so F_τ is not sensitive to initial conditions (same reasoning as in Proposition 2). Conversely, if τ cannot tile the plane, then there is n such that no valid tiling of a $(2n+1) \times (2n+1)$ square exists. This implies that, in any configuration x of F_τ, there is some z_0 with $\|z_0\|_\infty \leq n$ such that either $x(z_0) \in \mathcal{L}$, or $F_\tau(x)(z_0) \in \mathcal{L}$ (z_0 corresponds to some error for $\Sigma_{F,\tau}$). Then, applying Lemma 3 to position z_0 as in the proof of Proposition 3, we have:

$$\forall \delta > 0, \exists y, \exists t \geq 0 : d(x,y) \leq \delta \text{ and } d\big(F_\tau^t(x), F_\tau^t(y)\big) \geq 2^{-n}.$$

Since the constant n is independent of the choice of the initial configuration x, we have shown that F_τ is sensitive to initial conditions with sensitivity constant 2^{-n}. □

4.2 Controlling Erosion

In this section, we define G_τ, another variant of F, which has an overall similar behaviour but uses a different kind of obstacles and a different kind of erosion process depending on a tile set τ. Obstacles are protected from liquid component by a boundary as the classical obstacles of F, but they are made only of successive boundaries like onion skins. Moreover, invalid patterns in the solid component do not provoke the complete destruction of obstacles as in F.

The solid component of G_τ is the set $\mathcal{R}_\tau = \tau \times X$ where

$$X = \{\downarrow, \uparrow, \leftarrow, \rightarrow, \swarrow, \searrow, \nwarrow, \nearrow, \perp\}.$$

The liquid component is identical to that of F, precisely $\mathcal{L} = \{U, D, 0\}$.

The obstacle sub-shift $\Sigma_{G,\tau}$ of G_τ is defined by the set of allowed patterns constituted by all 3×3 patterns appearing in the following set of partial configurations:

$$\begin{array}{cccccccc}
\mathcal{L} & \mathcal{L} & \mathcal{L} & \mathcal{L} & \mathcal{L} & \mathcal{L} & \mathcal{L} & \mathcal{L} \\
\mathcal{L} & \mathcal{L} & \mathcal{L} & \mathcal{R}_\tau & \mathcal{R}_\tau & \mathcal{R}_\tau & \mathcal{L} & \mathcal{L} \\
\mathcal{L} & \mathcal{L} & \mathcal{L} & \mathcal{R}_\tau & \mathcal{R}_\tau & \mathcal{R}_\tau & \mathcal{L} & \mathcal{L} \\
\mathcal{L} & \mathcal{L} & \mathcal{L} & \mathcal{R}_\tau & \mathcal{R}_\tau & \mathcal{R}_\tau & \mathcal{L} & \mathcal{L} \\
\mathcal{L} & \mathcal{L} & \mathcal{L} & \mathcal{L} & \mathcal{L} & \mathcal{L} & \mathcal{L} & \mathcal{L}
\end{array}$$

with the additional conditions that the τ component is a valid tiling and the X component is made exclusively from the set of 2×2 patterns appearing in the following partial configuration:

$$\begin{array}{ccccccc}
 & & & \downarrow & \downarrow & & \\
 & \searrow & \downarrow & \downarrow & \downarrow & \swarrow & \\
\rightarrow & \rightarrow & \searrow & \downarrow & \swarrow & \leftarrow & \leftarrow \\
\rightarrow & \rightarrow & \rightarrow & \perp & \leftarrow & \leftarrow & \leftarrow \\
 & \rightarrow & \nearrow & \uparrow & \nwarrow & \leftarrow & \\
 & \nearrow & \uparrow & \uparrow & \uparrow & \nwarrow & \\
 & & & \uparrow & \uparrow & &
\end{array}$$

The X component is used to give to any cell inside a solid region a local notion of *inside* and *outside* as depicted by Figure 4 (up to $\pi/2$ rotations): arrows point to the inside region.

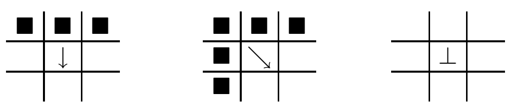

Fig. 4. Inside (white) and outside (black) positions for states of X.

The behaviour of G_τ is precisely the following:

1. if the neighbourhood (5×5 cells) forms a pattern forbidden in $\Sigma_{G,\tau}$, then the state is left unchanged except for the following cases where it turns into state 0:
   - if the cell is in a liquid state;
   - if the inside region of the cell forms a forbidden pattern,
   - the cell together with one of its neighbour forms a forbidden pattern
2. else, apply (if possible) one of the transition rules depending only on the 3×3 neighbourhood detailed in Figure 2 (replacing $\mathcal{S}$ by $\mathcal{R}_\tau$);
3. in any other case, leave the state unchanged if it is solid and turn into 0 if it is liquid.

As for F, a configuration is said *finite* if it contains only a finite number of cells in a solid state.

Lemma 4 (erosion result). *Let τ be any tile set. Let $x \in \Sigma_{G,\tau}$ be a finite configuration. Then the set $X = \{z \in \mathbb{Z}^2 : x(z) \in \mathcal{R}_\tau\}$ is a union of disjoint squares with sides of odd length containing the state '$\perp$' at the center, and which are pairwise spaced by at least 2 cells.*

Proof. By definition of $\Sigma_{G,\tau}$, x is necessarily made of rectangular obstacles which are pairwise spaced by at least 2 cells.

Moreover, the X component ensures that the border of any rectangular obstacle is made as follows:

- only state $\rightarrow$ (resp. $\leftarrow$, $\downarrow$ and $\uparrow$) on the left (resp. right, top and bottom) side;
- only state $\searrow$ (resp. $\swarrow$, $\nwarrow$ and $\nearrow$) on the top-left (resp. top-right, bottom-right and bottom-left) corner.

Remark that the X component requires that the sequence of state obtained by starting from a corner and advancing in the corresponding diagonal direction is a succession of identical diagonal arrows, then the state '$\perp$' and then a sequence of opposite diagonal arrows. This implies that the obstacle is a square of odd side length and that the state '$\perp$' is in the center. $\square$

From now on, we call *valid obstacle* for G_τ a $n \times n$ square (n odd) of solid states with state '$\perp$' in the center and forming a valid pattern of $\Sigma_{G,\tau}$.

Lemma 5 (conservative erosion process). *Let τ be any tile set. For any finite configuration x we have the following:*

1. *there exists t_0 such that, $\forall t \geq t_0$, $G_\tau^t(x) \in \Sigma_{G,\tau}$ and, in $G_\tau^t(x)$, any occurrence of U or D is on the left of any occurrence of any state from $\mathcal{R}_\tau$;*
2. *if z_0 and $n \geq 7$, n odd, are such that x contains a valid $n \times n$ obstacle centered on z_0 then $\forall t \geq 0$ $G_\tau^t(x)$ contains the same valid $(n-4) \times (n-4)$ square obstacle centered on z_0*

Proof. The first part of this lemma follows by applying arguments of the proof of Lemma 1 to G_τ. The only point to check is that given any forbidden pattern for $\Sigma_{G,\tau}$ we have (straightforward from the definition of $\Sigma_{G,\tau}$ and interior regions):

– either a pair of cells at distance at most 2, both in a solid state, and which form a forbidden pattern by themselves,
– or a cell in a solid state whose inside region forms a forbidden pattern.

Thus, the number of cells in a solid state is guaranteed to decrease while the current configuration is not in $\Sigma_{G,\tau}$. Therefore $\Sigma_{G,\tau}$ is reached in finite time (any configuration without solid states belongs to $\Sigma_{G,\tau}$).

For the second part of the lemma, consider all cells z of the lattice such that $\|z - z_0\|_\infty \leq \frac{n-5}{2}$ (i.e. cells belonging to the $(n-4) \times (n-4)$ square centered on z_0). Initially, those cells have a valid neighbourhood so after one step, they all stay in the same state. Therefore, by definition of a valid square obstacle, they all have a valid interior region after one step. Moreover, in their exterior region, they all have either valid solid states as in the initial step, or liquid states (if some cell at the boundary of the $n \times n$ square has turned into state 0): in any case, by definition of exterior regions, no such cell z has a cell in its neighbourhood to form a forbidden pattern with. Therefore, all cells z stay unchanged after two step, and the reasoning can be iterated forever. □

The infiltration lemma (Lemma 3 for F) remains true here, simply replacing $\Sigma_{\mathcal{S}}$ by $\Sigma_{G,\tau}$. Combined with the above lemmas, it implies the following proposition.

Proposition 5. *Let τ be any tile set. Then G_τ is sensitive to initial conditions if τ does not tile the plane, and it admits equicontinuous points if τ tiles the plane. Moreover, in the latter case, any equicontinuous point has the following properties:*

– *it is made only of solid states;*
– *it contains exactly one occurrence of state '$\perp$';*
– *its τ component forms a valid tiling.*

Proof. First, suppose that τ cannot tile the plane. Then there exists n such that there is no valid square tiling of size $n \times n$. Using the same reasoning as in Proposition 4, we deduce that G_τ is sensitive to initial conditions (because, by Lemma 5, after some time a liquid state must appear at some position z with $\|z\|_\infty \leq n$ and the infiltration can be applied to that position).

Now suppose that τ can tile the plane. Consider the configuration x made only of solid states and such that:

– the τ component is a valid tiling;
– the X component is made of state $\perp$ is at position $(0,0)$ and completed everywhere in a valid way.

Since any $n \times n$ square centered on position $(0,0)$ is a valid square obstacle, Lemma 5 shows that x is an equicontinuous point. Indeed, for any n and for any configuration y having a valid $n \times n$ square obstacle centered on position $(0,0)$, we have that the orbits of x and y under the action of G_τ coincide on the central $(n-4) \times (n-4)$ part.

Finally, consider any equicontinuous point x of G_τ. Using the reasoning of the first part of this proof, we show that x contains only solid states and that its τ component forms a valid tiling. Moreover, suppose that the X component contain at least 2 occurrences of state '$\perp$' and let n be such that 2 occurrences of '$\perp$' are contained in the $n \times n$ central square of x. By Lemmas 5 and 4, for any finite configuration y identical to x on the central $n \times n$ region, there is some time after which some cell in the central $n \times n$ region is in a liquid state (because no valid obstacle can contain two occurrences of '$\perp$'). From that point, the infiltration argument can be applied, contradicting the fact that x is an equicontinuous point. To conclude the proposition, it remains the case where the configuration x considered contains no occurrence of '$\perp$'. This case is treated as above, since valid square obstacles must contain an occurrence of '$\perp$' as stated by Lemma 4. □

4.3 Combining two solid components

Our last variation, called H_τ, is a simple combination of F and G_τ (for any given tile set τ). More precisely, it is the CA defined over state set $\mathcal{S} \cup \mathcal{R}_\tau \cup \mathcal{L}$ with the following behaviour:

- if the neighbour contains only states from $\mathcal{S} \cup \mathcal{L}$ then behave like F;
- if the neighbour contains only states from $\mathcal{R}_\tau \cup \mathcal{L}$ then behave like G_τ;
- in any other case, turn into state 0.

Using what was previously established for F and G_τ, we have the following proposition for H_τ.

Proposition 6. *Let τ be any tile set. Then H_τ is not sensitive to initial conditions and it admits equicontinuous points if and only if τ tiles the plane.*

Proof. Since arbitrarily large obstacles of type $\mathcal{S}$ can be formed, the reasoning of the proof of Proposition 2 can be applied here showing that H_τ is not sensitive to initial conditions.

Moreover, any equicontinuous point of x of G_τ is an equicontinuous point of H_τ. Indeed, for any n, any configuration y identical to x is the central $n \times n$ region verifies that at any time t, the central $n \times n$ region of $H_\tau^t(y)$ is made only of states from $\mathcal{R}_\tau \cup \mathcal{L}$ and is therefore governed by G_τ. Thus, the reasoning of Proposition 5 applies here. Hence, if τ can tile the plane, then H_τ admits equicontinuous points.

Conversely, suppose that τ cannot tile the plane. So H_τ has no equicontinuous points in $(\mathcal{R}_\tau \cup \mathcal{L})^{\mathbb{Z}^2}$ because it would be an equicontinuous point for G_τ, thus contradicting Proposition 5. Similarly, there cannot be equicontinuous point in

$(\mathcal{S} \cup \mathcal{L})^{\mathbb{Z}^2}$ because it would contradict Proposition 3. Finally, a configuration x containing states from both sets $\mathcal{S}$ and $\mathcal{R}_\tau$ cannot be an equicontinuous point either because x or $H_\tau(x)$ necessarily contains a liquid state and in such a case the infiltration argument can be applied as in Proposition 3 (Lemmas 2, 1 and 3 are true for H_τ). □

Automaton	*Solid component*	*Behaviour*
F	$\mathcal{S}$	$\mathcal{N}$
F_τ	$\mathcal{S}_\tau$	$\mathcal{S}_{ens}$ if τ tiles, $\mathcal{N}$ else
G_τ	$\mathcal{R}_\tau$	$\mathcal{E}_{qu}$ if τ tiles, $\mathcal{S}_{ens}$ else
H_τ	$\mathcal{R}_\tau \cup \mathcal{S}$	$\mathcal{E}_{qu}$ if τ tiles, $\mathcal{N}$ else

Fig. 5. Summary of constructions

5 Topological Classification Revisited

Equipped with the various constructions detailed above (see Figure 5), we study in this section the topological classification of P. Kůrka (put aside expansivity) for higher dimensional cellular automata.

In [6], the authors give a recursive construction which produce either a 1D CA with equicontinuous points or a 1D sensitive CA according to whether a Turing machine halts on the empty input or not. By Proposition 1, we get the following result.

Proposition 7. *For any dimension, the classes $\mathcal{S}_{ens}$ and $\mathcal{E}_{qu}$ are recursively inseparable. Moreover, $\mathcal{S}_{ens}$ is not recursively enumerable and $\mathcal{E}_{qu}$ is not co-recursively enumerable.*

However, this is not enough to establish the overall undecidability of the topological classification of 2D CA. The main concern of this section is to complete Proposition 7 in order to prove a stronger and more complete undecidability result summarised in the following theorem.

Theorem 1. *For any dimension strictly greater than* 1*, we have the following:*

- *each of the classes $\mathcal{E}_{qu}$, $\mathcal{S}_{ens}$ and $\mathcal{N}$ is neither recursively enumerable nor co-recursively enumerable;*
- *any pair of them is recursively inseparable.*

Proof. The proof of this theorem is made of 3 similar parts: each one gives the inseparability of two classes A and B among $\mathcal{S}_{ens}$, $\mathcal{E}_{qu}$ and $\mathcal{N}$, as well as the non enumerability of A and the non co-enumerability of B. The propositions focus on 2D cellular automata but, by Proposition 1, results remain true for higher dimensions (because the canonical lift from some CA F to $F^\uparrow$ is recursive). The 3 parts are proved in the following way:

$A = \mathcal{S}_{ens}$ and $B = \mathcal{E}_{qu}$: this is Proposition 7 (our construction G_τ gives an alternative proof by Berger's theorem).

$A = \mathcal{N}$ and $B = \mathcal{S}_{ens}$: this follows by Berger's theorem [10] (the set of tile sets which can tile the plane is not recursively enumerable) and Proposition 4 since F_τ can be recursively constructed from τ.

$A = \mathcal{E}_{qu}$ and $B = \mathcal{N}$: again since the set of tile sets that can tile the plane is not recursively enumerable, this follows by Proposition 6. □

Besides complexity of decision problems, other differences appears between dimension 1 and higher dimensions. Let us first stress the dynamical consequence of the construction of CA F_τ. It is well-known that for any 1D sensitive CA of radius r, 2^{-2r} is always the maximal admissible sensitivity constant (see for instance [2]). Thanks to the above construction it is easy to construct CA with tiny sensitivity constants as shown by the following proposition.

Proposition 8. *The (maximal admissible) sensitivity constant of sensitive 2D CA cannot be recursively (lower-)bounded in the number of states and the neighbourhood size.*

Proof. This follows directly from Proposition 4 since the size n of the largest $n \times n$ valid tiling for a given tile set is not a recursive function of the tile set. □

To finish this section, we will discuss another difference between 1D and 2D concerning the complexity of equicontinuous points. Let us first recall that equicontinuous point in 1D CA can be generated by finite words often called "blocking" words. A finite word u is blocking for some CA F if for any pair of configurations x and y both having pattern u in their center, we have[3]:

$$\forall t \geq 0, \forall z : \|z\|_\infty \leq r \Rightarrow F^t(x)(z) = F^t(y)(z)$$

where r is the radius of F.

For any F with equicontinuous points, there exists a finite word u such that ${}^\infty u^\infty$ is an equicontinuous point for F (proof in [2]). The construction G_τ can be used with the tile set of Myers [11] which can produce only non-recursive tilings of the plane. Therefore the situation is more complex in 2D, and we have the following proposition.

Proposition 9. *For any dimension strictly greater than* 1*, there exists a CA having equicontinuous points, but only non-recursive ones.*

[3] To simplify the definition, we require that the blocking word fixes the $2r+1$ central columns of the space-time diagrams of any configuration having u in its center. In fact $2r$ columns would be enough (and it is the standard definition) but it doesn't change anything for our purpose since with our definition of blocking word, we still have the property that a 1D CA admits equicontinuous points if and only if it has a blocking word.

Proof. By Proposition 5, any equicontinuous point of G_τ is made solely of solid states and its τ component forms a valid tiling. Now consider the tile set τ_0 of Myers [11]: it can tile the plane but only with non-recursive tilings. Therefore, by Proposition 5, G_{τ_0} admits equicontinuous points, but only non recursive ones □

Remark 1. Since the construction G_τ enforces the apparition of a particular state ($\perp$) in any equicontinuous point, we could have proved Proposition 9 using the simpler tile set of Hanf [12], which produces only non-recursive tilings provided some fixed tile is placed at the origin.

Any 1D CA with equicontinuous points, admits in fact uncountably many equicontinuous points. Indeed, if u is a blocking word and if c is any bi-infinite sequence of 0 or 1, then the configuration:

$$\cdots c(-n) \cdot u \cdots c(-1) \cdot u \cdot c(0) \cdot u \cdots c(n) \cdots$$

is always an equicontinuous point. The next proposition shows that it is no longer the case for higher dimensional CA.

Proposition 10. *For any dimension strictly greater than* 1*, there exists a CA having a countably infinite set of equicontinuous points.*

Proof. Let τ_0 be a trivial tile set (a single tile and no constraint). By Proposition 5, G_{τ_0} admits equicontinuous points which are all identical on their tiling component. Moreover, it follows from definition of Σ_{G,τ_0} that if two equicontinuous points have the state '$\perp$' in the same position, then they are identical. Thus G_{τ_0} possesses only a countable set of equicontinuous points and the proposition follows for dimension 2.

For dimension 3 we will use a lifted version G_+ of G_{τ_0}: G_+ is essentially a canonical lift of G_{τ_0} with the additional condition that 2 cells whose coordinates differ by 1 only on the third dimension must be in the same state, otherwise they turn into state 0 whatever the 2D dynamics of G_{τ_0} says. By a straightforward adaptation of the reasoning of Proposition 5 we have the following: for any equicontinuous point of G_+, the set of occurrences of states '$\perp$' is exactly a line co-linear to the third dimension. Therefore, by the same reasoning as above, we deduce that G_+ has only a countable set of equicontinuous points.

The lift arguments can be iterated and thus the proposition follows for any dimension. □

6 Complexity of Sensitivity According to Dimension

In this section, we study the complexity of the set of $\mathcal{S}_{ens}$ from the point of view of the arithmetical hierarchy. More precisely, we establish an upper bound in the 1D case and a lower bound in the 3D case showing that the complexity of $\mathcal{S}_{ens}$ does vary with dimension.

Proposition 11. *For 1D cellular automata, the set $\mathcal{S}_{ens}$ is Π^0_2.*

Proof. As said above, a 1D CA is sensitive if and only if it does not possess any blocking word [2]. Let F be a CA of radius r. Following the definition of blocking words given in Section 5, the fact that F possesses a blocking word can be expressed as follows:

$$\exists u \,\forall t \; R(u,t)$$

where $R(u,t)$ is true if and only if for all $t' \leq t$ and all pair of configurations x and y having u in their center, we have:

$$\forall z : \|z\|_\infty \leq r \Rightarrow F^{t'}(x)(z) = F^{t'}(y)(z).$$

$R(u,t)$ is recursive since the checking involve only a finite part of the initial configuration (precisely the $2r(t+1)$ central cells). Hence, the set $\mathcal{S}_{ens}$ is characterised by the Π^0_2 predicate $\forall u \,\exists t \;\neg R(u,t)$. $\square$

We will now give a hardness result for the set $\mathcal{S}_{ens}$ in dimension 3. We will reduce COFIN, the set of Turing machines halting on a co-finite set of inputs, to $\mathcal{S}_{ens}$ thus proving that $\mathcal{S}_{ens}$ is Σ^0_3-hard (see [13] for the proof of Σ^0_3-completeness of COFIN).

We will use simulations of Turing machines by tile sets in the classical way (originally suggested by Wang [14]): the tiling represents the space-time diagram of the computation and the transition rule of the Turing machine are converted into tiling constraints. For technical reasons which will appear clearly in the proof of Lemma 6, we slow down the computation (what can be done by a recursive modification of the machine): the head takes 2 time steps to move 1 cell left or right. Moreover, the tile sets we consider always contain some blank tile β (corresponding to a blank tape symbol of the Turing machine) and some special tile α used to initiate the computation, but *no tile corresponding to a final state of the Turing machine.* More precisely, each tile set enforces the following:

– if some row contains α, it is of the form ${}^\infty\beta\alpha w\beta^\infty$ where w is a sequence of non-blank symbols which will be treated as input (at this point we can not enforce by tiling constraint that w is finite);
– the tile on the right of α must represent the Turing head in its initial state reading the first letter of the input.

Thus, each time a valid tiling contains α, we are guaranteed that it contains a valid non-halting computation starting on some (potentially infinite) input.

The i^{th} Turing machine in a standard enumeration is denoted by $\mathcal{M}_i$ and to each $\mathcal{M}_i$ we associate a tile set τ_i whose constraints ensure the simulation of $\mathcal{M}_i$ as mentioned above, and which contains the special tiles α_i and β_i as described above.

We now describe the construction, for any Turing machine $\mathcal{M}_i$, of a cellular automaton I_i which is sensitive to initial conditions if and only if $\mathcal{M}_i \in$ COFIN. It will essentially consist in a lift to dimension 3 of a modified version of G_{τ_i}. We first describe this modified version, denoted $G_{<i>}$, which is a 2D CA.

The intuition is the following: we want that any equicontinuous point of $G_{<i>}$ contains a valid non-halting computation of $\mathcal{M}_i$ starting from a finite input. More precisely, we will define $G_{<i>}$ in such a way that any equicontinuous point has a valid τ_i-tiling on some of its components, which contains an occurrence of the special state α_i, and which contains only a *finite* sequence of non blank symbols on the right of α_i.

The definition of $G_{<i>}$ differ from that of G_{τ_i} only by the definition of the subshift Σ_{G,τ_i}: for $G_{<i>}$ this subshift becomes $\Sigma_{<i>}$ defined as follows. A configuration x is in $\Sigma_{<i>}$ exactly when:

- $x \in \Sigma_{G,\tau_i}$;
- α_i is the only tile allowed in the tiling component of a state having its X component equal to $\perp$;
- a solid state having a tile different from β_i in its tiling component is not allowed to be on the immediate left of a liquid state.

$G_{<i>}$ is built upon $\Sigma_{<i>}$ exactly as G_{τ_i} is built upon Σ_{G,τ_i}. Precisely, any cell of $G_{<i>}$ behave like this:

1. if the neighbourhood (5×5 cells) forms a pattern forbidden in $\Sigma_{<i>}$, then the state is left unchanged except in the following cases where it turns into state 0:
   - if the cell is in a liquid state;
   - if the inside region of the cell forms a forbidden pattern,
   - the cell together with one of its neighbour forms a forbidden pattern
2. else, apply (if possible) one of the transition rules depending only on the 3×3 neighbourhood detailed in Figure 2 (replacing $\mathcal{S}$ by $\mathcal{R}_{\tau_i}$);
3. in any other case, leave the state unchanged if it is solid and turn into 0 if it is liquid.

From this definition and the result already established for G_{τ_i} we easily get the following lemma.

Lemma 6. *$G_{<i>}$ is sensitive to initial conditions if $\mathcal{M}_i$ halts on any input. Moreover, if $\mathcal{M}_i$ doesn't halt on all inputs, then $G_{<i>}$ admits equicontinuous points and each equicontinuous point verifies the following:*

- *its tiling component forms a valid tiling for τ_i;*
- *it contains exactly one occurrence of the special tile α_i;*
- *there is a finite sequence w of consecutive non-blank symbols on the right of α_i, therefore the tiling component simulates a valid non-halting computation of $\mathcal{M}_i$ starting on a finite input w.*

Proof. The modifications introduced in $G_{<i>}$ (compared to G_{τ_i}) concern only new cases in which a solid state is turned into 0. Therefore, all necessary conditions about equicontinuous points of G_{τ_i} (Proposition 5) apply here. Besides, if $\mathcal{M}_i$ possesses a non-halting input, it is easy to construct an equicontinuous point x which contains a valid space-time diagram of a non-halting computation. The fact that the computation is slow ensures that that we can find arbitrarily large squares centered on the tile α_i (and the state $\perp$) without any non-blank on the right boundary of the square. With such precautions, the conservative erosion apply here exactly as in the proof of Proposition 5.

Finally, since the definition of $G_{<i>}$ implies that occurrences of $\perp$ coincide with occurrences of α_i, the lemma follows from the following property: if a configuration x of $G_{<i>}$ contains a cell having an infinite sequence of non-blank symbols on its right, then it is not an equicontinuous point. This property follows from the definition of $\Sigma_{<i>}$ since, for any finite configuration sufficiently close to x, the non-blank symbols allow liquid states to infiltrate towards a fixed position (after some time) and therefore the usual technique of particle infiltration shows that x cannot be an equicontinuous point. $\square$

The 3-dimensional cellular automaton I_i. The idea is that on each horizontal plane $\mathcal{P}_c = \{(a,b,c) : a,b \in \mathbb{Z}^2\}$ of the space, I_i generally behaves like $G_{<i>}$. However, I_i contains an additional 3D mechanism, whose role is to ensure that the non-halting simulations done on successive planes start from different inputs of $\mathcal{M}_i$. I_i contains an additional component of states, called Z, that can take 3 values '+', '−' and '=' (the state set of I_i is $Q_i \times Z$ where Q_i is the state set of $G_{<i>}$). To describe the local constraints on Z, we use notations $\texttt{South}\,(\cdot)$, $\texttt{North}\,(\cdot)$, $\texttt{East}\,(\cdot)$, $\texttt{West}\,(.)$ to describe relation between positions in the same horizontal plane, and $\texttt{Top}\,(\cdot)$ and $\texttt{Bot}\,(\cdot)$ for the 3^{rd} dimension:

- if the Z-component of a cell $z \in \mathbb{Z}^3$ is '=' then it is also the case for cells $\texttt{East}\,(z)$, $\texttt{West}\,(z)$, $\texttt{Top}\,(z)$ and $\texttt{Bot}\,(z)$;
- if the Z-component of a cell $z \in \mathbb{Z}^3$ is '+' then it is also the case for cells $\texttt{East}\,(z)$, $\texttt{West}\,(z)$ and $\texttt{Top}\,(z)$, whereas $\texttt{North}\,(z)$ and $\texttt{South}\,(z)$ must have a Z-component equal to '=';
- if the Z-component of a cell $z \in \mathbb{Z}^3$ is '−' then it is also the case for cells $\texttt{East}\,(z)$, $\texttt{West}\,(z)$ and $\texttt{Bot}\,(z)$, whereas $\texttt{North}\,(z)$ and $\texttt{South}\,(z)$ must have a Z-component equal to '=';
- if the tiling component of a cell z (in a solid state) is α_i then its Z-component must be either '+' or '−'; moreover $\texttt{Top}\,(z)$ and $\texttt{Bot}\,(z)$ must also be in a solid state with a tiling component equal to 'α_i';
- if a cell z in a solid state has its Z-component equal to '+' and its tiling component is β_i, then, if $\texttt{West}\,(\texttt{Bot}\,(z))$ has also its Z-component equal to '+' and is also solid, it must have its tiling component also equal to β_i;
- if a cell z in a solid state has its Z-component equal to '−' and its tiling component is β_i, then, if $\texttt{West}\,(\texttt{Top}\,(z))$ has also its Z-component equal to '+' and is also solid, it must have its tiling component also equal to β_i.

The global result of those local conditions is illustrated by the following lemma.

Lemma 7. *Let x be a purely solid configuration of I_i such that, each horizontal plane contains one occurrence of α_i and a valid tiling, and all the previous local conditions are verified. Then x has the following form:*

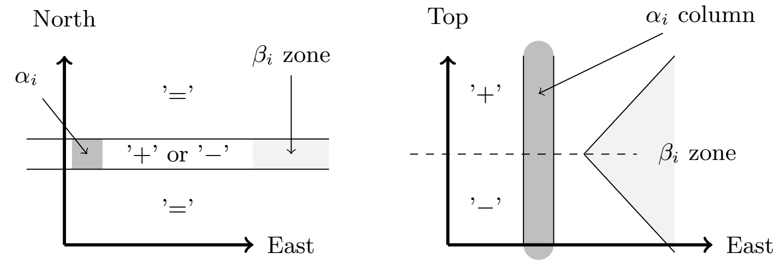

Fig. 6. Two planar (simplified) views of a valid solid configuration.

- *on each plane, all Z components are '$=$' except on an east/west line which contains α_i;*
- *all the occurrences of α_i are aligned in a top/bottom column;*
- *the space is made of a top half corresponding to planes $\mathcal{P}_c$ having some state with Z-component '$+$' and a bottom half corresponding to planes $\mathcal{P}_c$ having some state with Z-component '$-$';*
- *if a plane $\mathcal{P}_c$ is in the top half and simulates $\mathcal{M}_i$ on an input of length n, then for any $a > 0$, the plane $\mathcal{P}_{c+a}$ simulates $\mathcal{M}_i$ on an input of length strictly greater than n;*
- *similarly for the bottom half, the input length is strictly greater for plane $\mathcal{P}_{c-a}$ than for plane $\mathcal{P}_c$.*

Proof. Straightforward. □

I_i is then defined as follows: if one of the previous local conditions is violated in the neighbourhood of a cell in a solid state surrounded only by cells in a solid state, then the cell turns into state $(0, =)$, else it behaves according to $G_{<i>}$ depending only on cells in the same plane.

Proposition 12. *For dimension 3, the set $\mathcal{S}_{ens}$ is Σ_3^0-hard.*

Proof. We show that I_i is sensitive to initial conditions if and only if $\mathcal{M}_i$ admits an infinite set of non-halting inputs, which yields a reduction from COFIN to $\mathcal{S}_{ens}$.

First, it is easy to see that if $\mathcal{M}_i$ has an infinite set of non-halting inputs, then an equicontinuous point for I_i can be build: given an infinite sequence of non-halting inputs of different lengths, one can build a purely solid configuration, made of two halves, each one corresponding to the sequence of valid simulations on each plane for successive inputs, and respecting all the conditions on the Z component. It is straightforward to check that such a configuration is an equicontinuous point.

Conversely, if x is an equicontinuous point for I_i then each plane $\mathcal{P}_c$ must be an equicontinuous point for $G_{<i>}$ when we forget the Z component. Indeed, the additional 3D conditions of I_i never affect liquid states and can only turn a solid state into state 0. Now, adding 3D constraints, we deduce by Lemmas 6 and 7 that $\mathcal{M}_i$ must have an infinite set of non-halting inputs. □

7 Future Work

In this paper, we adopted the classical framework of topological dynamics (which does not explicitly refer to dimension) and studied how its application to cellular automata may vary with dimension.

The first research direction opened by this paper is the study of new dynamical behaviour appearing in dimension 2 and more. Indeed, the mechanisms of information propagation can no longer be explained by the presence of particular finite words (blocking words in dimension 1). In this general direction, the following questions seems particularly relevant to us:

- what kind of dynamics can be found in the class $\mathcal{N}$?
- what kind of 2D cellular automata can be built which are in $\mathcal{E}_{qu}$ and have a set of equicontinuous points of full measure? can we characterise such CA?
- what happens when we restrict to reversible cellular automata? more generally to surjective ones?

The second part of the paper concerns complexity of decision problems related to topological dynamics properties. Our construction techniques allow to prove several complexity lower bounds. However, upper bounds seems harder to establish. We think the following questions are worth being investigated:

- what is the exact complexity of $\mathcal{S}_{ens}$ in 1D? is it Π_2-complete or only at level 1 of the arithmetical hierarchy?
- we believe that the set $\mathcal{S}_{ens}$ is in the arithmetical hierarchy for any dimension, but we have no proof yet starting from dimension 2.
- can we generally implement "Turing-jumps" in the complexity of the problem we consider when we increase dimension? or is there limitation coming from the nature of the problem?

Finally, the various kind of sensitivity to dimension change we encountered, suggest to consider those problems from of more general point of view by allowing the lattice of cells to be any Cayley graph. Can we then characterise graphs for which $\mathcal{S}_{ens}$ and $\mathcal{E}_{qu}$ are complementary classes? What can be said on the complexity of the different classes of topological dynamics?

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